Variational principle for Gibbs point processes with finite range interaction

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June 17, 2015

Abstract

The variational principle for Gibbs point processes with general finite range interaction is proved. Namely, the Gibbs point processes are identified as the minimizers of the free excess energy equals to the sum of the specific entropy and the mean energy. The interaction is very general and includes superstable pairwise potential, finite or infinite multibody potential, geometrical interaction, hardcore interaction. The only restrictive assumption involves the finite range property.

Keywords. specific entropy ; pairwise potential ; Strauss model ; Quermass-interaction

1 Introduction

Gibbs point processes are popular models to describe the repartition of points or geometrical structures in space. They appeared first for modelling continuum interacting particles in statistical mechanics. Now they are widely used in as different domains as astronomy, biology, computer science, ecology, forestry, image analysis, materials science. The main reason is that they provide a clear interpretation of the interactions between the points, such as attraction or repulsion depending on their interdistance. We refer to Preston (1976), Chiu et al. (2013), and Van Lieshout (2000) for classical text books on Gibbs point processes, including examples and applications.

The Gibbs point processes are defined via their local unnormalized conditional densities of the form e^{-H} where H is an energy functional. They are the equilibrium states of the DLR equations (see definition 2). A variational principle, coming from the statistical physics, claims that the Gibbs measures are also the minimizers of the free excess energy equals to the entropy plus the mean energy. More precisely, for any stationary probability measure P on the space of configurations in $\mathbb{R}^d$, the specific entropy $\mathcal{I}(P)$ with respect to the Poisson point process and the mean energy per unit volume $H(P)$ are defined via thermodynamic limits (see (5) and (7)). The principle claims that the Gibbs measures are exactly the probability measures P which minimize the functional $P \mapsto \mathcal{I}(P) + H(P)$. It thus supports the common belief that the Gibbs measures provide a proper description of physical systems in thermodynamic equilibrium. They are many applications of the variational principle in physics and mathematics. In statistical mechanics, the phase transition phenomenon (non uniqueness of Gibbs measures) can be proved in studying the geometry of the set of Gibbs measures and in particular its extremal points. The variational principle is a key tool in this study (see Georgii (2011) for a general presentation). In probability theory, it is related to the large deviation principle for the empirical field Georgii (1994b). In spatial statistic, it is a crucial identifiability assumption for the consistency of the maximum likelihood estimator Dereudre and Lavancier (2015). This last recent paper highlights the importance of the variational principle for models coming from the spatial statistic. It was our initial motivation for the present paper.

For the lattice Gibbs models, the variational principle is well established and a general proof can be found in Preston (1976), Section 7. The first proof was for the Ising model in Landford and Ruelle (1969). In the setting of Gibbs point processes, they are less results. In Georgii (1994b,a) the author proves the variational principle for the pairwise potential energy $H(\omega) = \sum_{x,y} \phi(|x-y|)$ where the sum is over all couples $\{x, y\}$ in the configuration ω. The potential ϕ is assumed to be non-integrably divergent at the origin (i.e. $\int_0^1 \phi(r) r^{d-1} ds = \infty$) producing a strong repulsion when the particles are closed to each other. A typical examples is the Lennard Jones pairwise potential $\phi(r) = ar^{-12} - br^{-6}$. In another work the variational principle is proved for the Delaunay-tile interaction Dereudre and Georgii (2009). The energy function has the following form $H(\omega) = \sum_T \phi(T)$ where the sum is over all triangles T of the Delaunay triangulation based on ω. It is a continuum spatial version of nearest neighbours interaction models. As fas as we know both papers are the only ones proving the variational principle for Gibbs point processes models. Unfortunately many interesting energy functions are not covered by these results, as for example any pairwise potential energy with bounded potential ϕ. In particular, the well-studied Strauss model in spatial statistics with the pairwise potential $\phi(r) = \mathbb{1}_{[0,R]}(r)$ is uncovered. In stochastic geometry, the Area-interaction or the Quermass-interaction are not covered as well (see Baddeley and Lieshout (1995) and Kendall et al. (1999)).

In this paper we prove the variational principle for Gibbs point processes with general finite range interaction. The other assumptions are standard and satisfied by all models we met in statistical mechanics and spatial statistic (see Section 3). In particular, our setting includes superstable pairwise potential, finite or infinite multibody potential, geometrical interaction, hardcore interaction. The proof is based on fine controls of the relative entropy of P_Λ with respect to the Gibbs measure on Λ, where P is any stationary field on $\mathbb{R}^d$ and Λ an observable window tending to $\mathbb{R}^d$. This strategy was already present in Preston (1976), Georgii (1994a) and Georgii (1994b).

The paper is organized as follows. In section 2, we introduce the notations and the Gibbs models. The variational principle and the main theorem are presented in Section 3. Two standard examples are given in Section 4; the superstable pairwise potential with compact support and the Quermass interaction. Section 4 is devoted to the proof of our main theorem.

2 The Gibbs models

2.1 State spaces and reference measures

Our setting is the Euclidean space $\mathbb{R}^d$ of arbitrary dimension $d \geq 1$ equipped with its Borel σ-field. An element of $\mathbb{R}^d$ is denoted by x and the Lebesgue measure on $\mathbb{R}^d$ is denoted by λ^d. A **configuration** is a subset ω of $\mathbb{R}^d$ which is locally finite, meaning that $\omega \cap \Lambda$ has finite cardinality $N_\Lambda(\omega) = \#(\omega \cap \Lambda)$ for every bounded Borel set Λ. The space Ω of all configurations is equipped with the σ-algebra $\mathcal{F}$ generated by the counting variables N_Λ. The space of finite configurations is denoted by Ω_f.

The symbol Λ will always refer to a bounded Borel set in $\mathbb{R}^d$. It will often be convenient to write ω_Λ in place of $\omega \cap \Lambda$. We abbreviate $\omega \cup \{x\}$ to $\omega \cup x$ and abbreviate $\omega \backslash \{x\}$ to $\omega \backslash x$ for every ω and every x in ω.

As usual, we take the reference measure on $(\Omega, \mathcal{F})$ to be the distribution π of the Poisson point process with intensity measure λ^d on $\mathbb{R}^d$. Recall that π is the unique probability measure on $(\Omega, \mathcal{F})$ such that the following hold for all subsets Λ: (i) N_Λ is Poisson distributed with parameter $\lambda^d(\Lambda)$, and (ii) conditional on $N_\Lambda = n$, the n points in Λ are independent with uniform distribution on Λ. The Poisson point process restricted to Λ will be denoted π_Λ.

Translation by a vector $u \in \mathbb{R}^d$ is denoted by τ_u, either acting on $\mathbb{R}^d$ or on Ω. A probability P on Ω is said stationary if $P = P \circ \tau_u^{-1}$ for any u in $\mathbb{R}^d$. In this paper we consider only stationary probability measures P with finite finite intensity measure (i.e. $\mathbf{E}_P(N_{[0,1]^d}) < +\infty$). We denote by $\mathcal{P}$ the space of such probability measures.

2.2 Gibbs point processes models

We consider a measurable function H from Ω_f to $\mathbb{R} \cup \{+\infty\}$ which is called energy function. We assume that H is **stationary**; for any $\omega \in \Omega_f$ and any $u \in \mathbb{R}^d$, $H(\omega) = H(\tau_u(\omega))$. We assume also that the energy function H is **hereditary** which means that for any x in $\mathbb{R}^d$ and ω in Ω_f, $H(\omega \cup \{x\}) = +\infty$ as soon as $H(\omega) = +\infty$. The energy H is said **non-degenerate** if $H(\{0\}) \neq \infty$ and $H(\emptyset) = 0$. We assume also that H is **stable** which means that there exists $A > 0$ such that for any finite configuration $\omega \in \Omega_f$

$$H(\omega) \geq -AN(\omega). \tag{1}$$

All these assumptions are standard and non restrictive. The main restriction in the present paper is the **finite range** assumption which means that there exists $R \geq 0$ such that for any configuration ω, any bounded set Λ the quantity

$$H_\Lambda(\omega) := H(\omega_{\Lambda'}) - H(\omega_{\Lambda' \setminus \Lambda}) \tag{2}$$

(with the convention $\infty - \infty = 0$) does not depend on the choice of Λ' as soon as $\Lambda \oplus B(0, R) \subset \Lambda'$. $H_\Lambda(\omega)$ represents the energy of ω_Λ inside Λ given the configuration ω_{Λ^c} outside Λ.

The Gibbs measures P associated to H are defined through their local conditional specification, as described below. We denote by Ω_∞ the set of configurations $\omega \in \Omega$ such that for any Λ, $H(\omega_\Lambda) < +\infty$. So for every Λ and every configuration $\omega \in \Omega_\infty$, the local conditional density f_Λ of P with respect to π_Λ is defined by

$$f_\Lambda(\omega) \quad = \quad \frac{1}{Z_\Lambda(\omega_{\Lambda^c})} e^{-H_\Lambda(\omega)}, \tag{3}$$

where $Z_\Lambda(\omega_{\Lambda^c})$ is the normalization constant given by

$$Z_\Lambda(\omega_{\Lambda^c}) = \int e^{-H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} \pi_\Lambda(d\omega'_\Lambda).$$

Let us note that $0 < Z_\Lambda(\omega_{\Lambda^c}) < +\infty$ since H is stable and non-degenerate.

We are now in position to define the Gibbs measures associated to H (See Preston (1976) for instance).

Definition 1 *A probability measure P on Ω is a* **Gibbs measure** *for the energy function H if $P(\Omega_\infty) = 1$ and if for every bounded borel set Λ, for any measurable and bounded function g from Ω to $\mathbb{R}$,*

$$\int g(\omega) P(d\omega) = \int \int g(\omega'_\Lambda \cup \omega_{\Lambda^c}) f_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c}) \pi_\Lambda(\mathrm{d}\omega'_\Lambda) P(d\omega). \tag{4}$$

Equivalently, for P-almost every ω the conditional law of P given ω_{Λ^c} is absolutely continuous with respect to π_Λ with the density f_Λ defined in (3).

The equations (4) are called the Dobrushin–Lanford–Ruelle (DLR) equations. The existence of such Gibbs measures, in the present setting of finite range stable interactions, is done in Dereudre et al. (2012), Corollary 3.4 and Remark 3.1. Note that the uniqueness of such P does not necessarily hold, leading to the phase transition phenomenon. We denote by $\mathcal{G}_H$ the set of all Gibbs measures for the energy H.

3 Variational Principle

The variational principle in statistical mechanics claims that the Gibbs measures are the minimizers of the free excess energy defined by the sum of the the mean energy and the specific entropy. Moreover the minimum is equal to minus the pressure. Let us first define precisely all these

macroscopic quantities. For the sake of simplicity we consider the macroscopic limit along the sequence of sets $\Lambda_n = [-n,n]^d$, $n \geq 1$. Limits $\Lambda \to \mathbb{R}^d$ in the Van-Hove sens could have been considered as well.

Let P be a stationary probability measure in $\mathcal{P}$. The **specific entropy** of P is defined as the limit

$$\mathcal{I}(P) = \lim_{n\to+\infty} \frac{1}{|\Lambda_n|}\mathcal{I}(P_{\Lambda_n}, \pi_{\Lambda_n}), \tag{5}$$

where for any probability measures μ and ν

$$\mathcal{I}(\mu,\nu) = \begin{cases} \int \ln(f)d\mu & \text{if } \mu \ll \nu \text{ with density } f \\ +\infty & \text{otherwise} \end{cases}.$$

Note that the limit in (5) always exists; see Georgii (2011) for general results on specific entropy.

Let us now introduce the **Pressure**. It is defined as the following limit.

$$p_H := \lim_{n\to+\infty} \frac{1}{|\Lambda_n|} \ln(Z_n), \tag{6}$$

where $Z_n = Z_{\Lambda_n}(\emptyset)$ is the partition function with empty boundary condition.

In the following lemma we show that p_H always exists in the setting of the present paper.

Lemma 1 *Assuming that the energy function H is finite range, stable and non-degenerate, then the pressure p_H defined in* (6) *exists and belongs to* $[-1,(e^A-1)]$.

Proof.

For any set Λ we denote by $\Lambda^\ominus$ the set

$$\Lambda^\ominus = \{x\in\Lambda, B(x,R_0)\subset\Lambda\},$$

where R_0 is an integer larger than the range of the interaction R. So for $n > R_0$, $\Lambda_n^\ominus = \Lambda_{n-R_0}$.

For any $R_0 < m < n$, we consider the Euclidean division $n = km + l$ with $0 \leq l < m$, $k \geq 0$. Let $(\Lambda_m^i)_{1\leq i\leq k^d}$ a family of k^d disjoint cubes inside Λ_n where each cube is a translation of Λ_m.

From the definition of the partition function

$$\begin{aligned} Z_n &\geq \pi_{\Lambda_n}(\omega_{\Lambda_n\setminus\cup_i\Lambda_m^{i,\ominus}} = \emptyset)\int e^{-H(\omega)}\pi_{\cup_i\Lambda_m^{i,\ominus}}(d\omega) \\ &= e^{-(|\Lambda_n|-k^d|\Lambda_m^\ominus|)}\prod_i Z_{\Lambda_m^{i,\ominus}}(\emptyset) \\ &\geq e^{-(k^d 2dR_0(2m)^{d-1}+2dm(2n)^{d-1})} Z_{m-R_0}^{k^d}. \end{aligned}$$

So since $|\Lambda_n|/k^d$ goes to $|\Lambda_m|$ when n goes to infinity,

$$\liminf_{n\to\infty} \frac{1}{|\Lambda_n|}\ln(Z_n) \geq \frac{1}{|\Lambda_m|}\big(Z_{m-R_0} - 2dR_0(2m)^{d-1}\big).$$

This inequality holds for each $m \geq R_0$. So, letting m tends to infinity

$$\liminf_{n\to\infty} \frac{1}{|\Lambda_n|}\ln(Z_n) \geq \limsup_{m\to\infty}\frac{1}{|\Lambda_m|}Z_{m-R_0} = \limsup_{m\to\infty}\frac{1}{|\Lambda_m|}Z_m$$

which proves that the limit exists in $\mathbb{R}\cup\{\pm\infty\}$.

Thanks to the stability and the non degeneracy of H we get that

$$e^{-|\Lambda_n|} \leq Z_n \leq e^{|\Lambda_n|(e^A-1)}$$

which implies that $p_H \in [-1, (e^A - 1)]$.

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The last macroscopic quantity involves the mean energy of a stationary probability measure P. It is also defined by a limit but, in opposition to the other macroscopic quantities, we have to assume that it exists. The proof of such existence is based on stationary arguments and a nice representation of the energy contribution per unit volume. Examples are given in Section 4. So for any stationary probability measure P we assume that the following limit exists

$$H(P) := \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int H(\omega_{\Lambda_n}) P(d\omega). \tag{7}$$

and we call the limit **mean energy** of P.

We need to introduce a last technical assumption on the boundary effects of H. We assume that for any P in $\mathcal{G}_H$

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int \partial H_{\Lambda_n}(\omega) P(d\omega) = 0, \tag{8}$$

where $\partial H_{\Lambda_n}(\omega) = H_{\Lambda_n}(\omega) - H(\omega_{\Lambda_n})$. This assumption is satisfied by all the examples we meet.

Theorem 1 *We assume that H is stationary, hereditary, non-degenerate, stable and finite range. Moreover we assume that the mean energy exists for any stationary probability measure P (i.e. the limit* (7) *exists) and that the boundary effects assumption* (8) *holds. Then for any stationary probability measure $P \in \mathcal{P}$*

$$I(P) + H(P) \geq -p_H, \tag{9}$$

with equality if and only if P is a Gibbs measure (i.e. $P \in \mathcal{G}_H$).

4 Examples

In this section we present two examples of energy functions included in the setting of Theorem 1. The first example is the standard superstable pairwise potential energy. The second example involves the Quermass interaction which is an energy function for morphological patterns built by unions of random convex sets. It can be also viewed as a infinite body potential interaction. The main restriction in Theorem 1 is the finite range property and so any standard examples, having this property, could have been considered as well.

4.1 Pairwise potential

In this section the energy function has the following expression: for any finite configuration $\omega \in \Omega_f$

$$H(\omega) = zN(\omega) + \sum_{\{x,y\}\subset\omega} \phi(x-y), \tag{10}$$

where ϕ is a symmetric function from $\mathbb{R}^d$ to $\mathbb{R} \cup \{+\infty\}$ with compact support. The parameter $z > 0$ is called activity and allow to change the intensity of the reference Poisson point process. The potential ϕ is said stable if the associated energy H in (10) is stable. In the following we need that the potential ϕ is **superstable** which means that ϕ is the sum of stable potential and a positive potential which is non negative around the origin. See Ruelle (1969) for examples of stable and superstable pairwise potentials. In this setting the variational principle holds as a corollary of Theorem 1.

Corollary 1 *Let H be a energy function coming from a superstable pairwise potential ϕ given by* (10)*. Then for any stationary probability measure $P \in \mathcal{P}$*

$$I(P) + H(P) \geq -p_H, \tag{11}$$

with equality if and only if P is a Gibbs measure (i.e. $P \in \mathcal{G}_H$). The expression of $H(P)$ is given in (12)*.*

Proof.

Let us check the assumptions of Theorem 1. It is obvious that H is stationary, hereditary, non degenerate, stable and finite range. The existence of the mean energy is proved in Georgii (1994b), Theorem 1. It is given by

$$H(P) = \begin{cases} \frac{1}{2}\int \sum_{0 \neq x \in \omega} \phi(x) P^0(d\omega) & \text{if } E_P(N^2_{[0,1]^d}) < \infty \\ +\infty & \text{otherwise} \end{cases} \tag{12}$$

where P^0 is the Palm measure of P. Recall that P^0 can be viewed as the natural version of the conditional probability $P(.|0 \in \omega)$ (see Matthes et al. (1978) for more details). So, it remains to prove the boundary assumption (8). Let P a Gibbs measure in $\mathcal{G}_H$. A simple computation gives that for any $\omega \in \Omega$

$$\partial H_{\Lambda_n(\omega)} = \sum_{x \in \omega_{\Lambda_n^{\oplus} \setminus \Lambda_n}} \sum_{y \in \omega_{\Lambda_n \setminus \Lambda_n^{\ominus}}} \phi(x-y),$$

where $\Lambda_n^{\oplus} = \Lambda_{n+R_0}$ and $\Lambda_n^{\ominus} = \Lambda_{n-R_0}$ with R_0 an integer larger than the range of the interaction R. Using the GNZ equation (see Nguyen and Zessin (1979)), the stationarity of P we obtain

$$\begin{aligned} |E_P(\partial H_{\Lambda_n})| &\leq \int \sum_{x \in \omega_{\Lambda_n^{\oplus} \setminus \Lambda_n}} \sum_{y \in \omega \setminus x} |\phi(x-y)| P(d\omega) \\ &= \int \int_{\Lambda_n^{\oplus} \setminus \Lambda_n} e^{-z - \sum_{y \in \omega} \phi(x-y)} \sum_{y \in \omega} |\phi(x-y)| dx P(d\omega) \\ &= |\Lambda_n^{\oplus} \setminus \Lambda_n| e^{-z} \int e^{-\sum_{y \in \omega_{B(0,R_0)}} \phi(y)} \sum_{y \in \omega_{B(0,R_0)}} |\phi(y)| P(d\omega). \end{aligned}$$

Since ϕ is stable we deduce that $\phi \geq -A-2z$. So denoting by $C := \sup_{c \in [-A-2z; +\infty)} |c| e^{-c} < \infty$ we find that

$$|E_P(\partial H_{\Lambda_n})| \leq |\Lambda_n^{\oplus} \setminus \Lambda_n| C e^{-z} \int N_{B(0,R_0)}(\omega) e^{(A+2z) N_{B(0,R_0)}(\omega)} P(d\omega). \tag{13}$$

Using the estimates in Ruelle (1970) corollary 2.9, the integral in the right term of (13) is finite. The boundary assumption (8) follows.

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4.2 Quermass interaction

The Quermass process is a morphological interacting model introduced in Kendall et al. (1999) which is a generalization of the well-known Widom-Rowlinson process or Area Process (see Widom and Rowlinson (1970), Baddeley and Lieshout (1995)). Since the existence of the Quermass process is only proved

in the case $d \leq 2$ we restrict the following to the non trivial case $d = 2$. For any finite configuration ω, $L(\omega)$ denotes the set $\cup_{x\in\omega} B(x,r)$ and the energy is defined as a linear combination of the Minkowski functionals;

$$H(\omega) = \theta_1 \mathcal{A}\Big(L(\omega)\Big) + \theta_2 \mathcal{L}\Big(L(\omega)\Big) + \theta_3 \chi\Big(L(\omega)\Big), \tag{14}$$

where $r > 0$, $\theta_i \in \mathbb{R}$, $i = 1 \ldots 3$ are parameters and $\mathcal{A}$, $\mathcal{L}$, χ are respectively the area, the perimeter and the Euler-Poincaré characteristic functionals. Recall that $\chi(L(\omega))$ is equal to $N_{cc}(L(\omega)) - N_h(L(\omega))$ where $N_{cc}(L(\omega))$ denotes the number of connected components in $L(\omega)$ and $N_h(L(\omega))$ the number of holes. We refer to Chiu et al. (2013) for more details about Minkowski functionals.

Corollary 2 *Let H be the Quermass interaction given in* (14)*. Then for any stationary probability measure $P \in \mathcal{P}$*

$$I(P) + H(P) \geq -p_H, \tag{15}$$

with equality if and only if P is a Gibbs measure (i.e. $P \in \mathcal{G}_H$). The expression of $H(P)$ is given in (19)*.*

Proof.

As in the previous section, we check the assumptions of Theorem 1. It is obvious that H is stationary, hereditary and non degenerate. In dimension $d = 2$ the functional χ satisfied the following bound

$$|\chi(L(\omega))| \leq 3N(\omega), \tag{16}$$

see Kendall et al. (1999). The stability of H follows easily. The finite range assumption is a consequence of the additivity of Minkowski functionals. Note that the range of the interaction is $R = 2r$. In the following we denote by C the cube $[0,1]^2$, by ∂C the boundary of C and by $\hat{C}$ the double edges $\{0\} \times [0,1] \cup [0,1] \times \{0\}$. For any $k \in \mathbb{Z}^2$ we consider also the translations $C_k = \tau_k(C)$, $\partial C_k = \tau_k(\partial C)$ and $\hat{C}_k = \tau_k(\hat{C})$. Thanks to the additivity of Minkowski functionals we obtain that for any finite configuration ω and any $n \geq 1$

$$\begin{aligned} H(\omega_{\Lambda_n}) \quad = \quad & \sum_{k \in \{-n, n-1\}^2} \Bigg(\theta_1 \mathcal{A}\big(L(\omega) \cap C_k\big) + \theta_2 \Big[\mathcal{L}\big(L(\omega) \cap C_k\big) - \mathcal{L}\big(L(\omega) \cap \partial C_k\big) \Big] \\ & + \theta_3 \Big[\chi\big(L(\omega) \cap C_k\big) - N_{cc}\big(L(\omega) \cap \hat{C}_k\big) \Big] \Bigg) + R_n(\omega_{\Lambda_n}), \end{aligned} \tag{17}$$

which gives the energy contribution of each cube C_k in $H(\omega_{\Lambda_n})$. Thanks to (16) and obvious bounds for $\mathcal{A}$ and $\mathcal{L}$, the boundary term $R_n(\omega_{\Lambda_n})$ satisfies for some constant $c > 0$

$$|R_n(\omega_{\Lambda_n})| \leq cN\left(\omega_{\Lambda_n \setminus \Lambda_{n-R_0}}\right). \tag{18}$$

For any stationary probability measure $P \in \mathcal{P}$ we deduce easily from (17) and (18) the existence of the mean energy (7) with

$$H(P) = \int \theta_1 \mathcal{A}\big(L(\omega) \cap C\big) + \theta_2 \Big[\mathcal{L}\big(L(\omega) \cap C\big) - \mathcal{L}\big(L(\omega) \cap \partial C\big) \Big] + \theta_3 \Big[\chi\big(L(\omega) \cap C\big) - N_{cc}\big(L(\omega) \cap \hat{C}\big) \Big] P(d\omega). \tag{19}$$

Thanks to (17) and (18), the boundary assumption (8) is satisfied as well.

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5 Proof of Theorem 1

Let us start by proving the inequality (9) for any stationary probability measure $P \in \mathcal{P}$. For $n \geq 1$ we define the Gibbs measure on Λ_n with free boundary condition by

$$Q_n(d\omega_{\Lambda_n}) = \frac{1}{Z_n} e^{-H(\omega_{\Lambda_n})} \pi_{\Lambda_n}(d\omega_{\Lambda_n}). \tag{20}$$

So

$$\begin{aligned} \mathcal{I}(P_{\Lambda_n}, Q_n) &= \int \ln\left(\frac{dP_{\Lambda_n}}{dQ_n}(\omega_{\Lambda_n}) \right) dP_{\Lambda_n}(\omega_{\Lambda_n}) \\ &= \int \ln\left(\frac{dP_{\Lambda_n}}{d\pi_{\Lambda_n}}(\omega_{\Lambda_n}) \frac{d\pi_{\Lambda_n}}{dQ_n}(\omega_{\Lambda_n}) \right) dP_{\Lambda_n}(\omega_{\Lambda_n}) \\ &= \mathcal{I}(P_{\Lambda_n}, \pi_{\Lambda_n}) + \int H(\omega_{\Lambda_n}) dP(\omega) + \ln(Z_n), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathcal{I}_{\Lambda_n}(P_{\Lambda_n}, Q_n) = \mathcal{I}(P) + H(P) + p_H. \tag{21}$$

Since $\mathcal{I}_{\Lambda_n}(P_{\Lambda_n}, Q_n)$ is positive the inequality (9) follows. Let us now prove that for any $P \in \mathcal{G}_H$ the equality holds in (9). Let us show that the limit in (21) is negative. Recall that R_0 is an integer larger than the range of the interaction R and that $\Lambda_n^\oplus$ stands for the set Λ_{n+R_0}. We denote by $\pi_{\Lambda_n} \otimes P_{\Lambda_n^\oplus \setminus \Lambda_n}$ the law of the point process on $\Lambda_n^\oplus$ with independent configurations on Λ_n and $\Lambda_n^\oplus \setminus \Lambda_n$ with distributions π_{Λ_n} and $P_{\Lambda_n^\oplus \setminus \Lambda_n}$ respectively. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathcal{I}(P_{\Lambda_n}, Q_n) &= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathcal{I}(P_{\Lambda_n^\oplus}, Q_{n+R_0}) \\ &= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int \ln\Bigg(\frac{dP_{\Lambda_n^\oplus}}{d\pi_{\Lambda_n} \otimes P_{\Lambda_n^\oplus \setminus \Lambda_n}}(\omega_{\Lambda_n^\oplus}) \frac{d\pi_{\Lambda_n} \otimes P_{\Lambda_n^\oplus \setminus \Lambda_n}}{d\pi_{\Lambda_n^\oplus}}(\omega_{\Lambda_n^\oplus}) \\ &\qquad \frac{d\pi_{\Lambda_n^\oplus}}{dQ_{n+R_0}}(\omega_{\Lambda_n^\oplus}) \Bigg) dP_{\Lambda_n^\oplus}(\omega_{\Lambda_n^\oplus}) \\ &= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \Bigg(\mathcal{I}(P_{\Lambda_n^\oplus \setminus \Lambda_n}, \pi_{\Lambda_n^\oplus \setminus \Lambda_n}) + \ln(Z_{n+R_0}) + \int H(\omega_{\Lambda_n^\oplus}) - H_{\Lambda_n^\oplus}(\omega_{\Lambda_n^\oplus}) \\ &\qquad - \ln(Z_{\Lambda_n}(\omega_{\Lambda_n^\oplus})) dP_{\Lambda_n^\oplus}(\omega_{\Lambda_n^\oplus}) \Bigg), \end{aligned} \tag{22}$$

where the densities which appear above are given by (4) and (20). By subadditivity of the entropy (Proposition 15.10 in Georgii (2011)),

$$0 \leq \mathcal{I}(P_{\Lambda_n^\oplus \setminus \Lambda_n}, \pi_{\Lambda_n^\oplus \setminus \Lambda_n}) \leq \mathcal{I}(P_{\Lambda_n^\oplus \setminus \Lambda_n}, \pi_{\Lambda_n^\oplus \setminus \Lambda_n}) - \mathcal{I}(P_{\Lambda_n}, \pi_{\Lambda_n}),$$

which implies that

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathcal{I}(P_{\Lambda_n^\oplus \setminus \Lambda_n}, \pi_{\Lambda_n^\oplus \setminus \Lambda_n}) = 0.$$

Moreover, thanks to the existence of the mean energy and the boundary assumption assumption (8), the term $\lim_{n\to\infty} |\Lambda_n|^{-1} \int (H(\omega_{\Lambda_n^\oplus}) - H_{\Lambda_n^\oplus}(\omega) dP(\omega)$ vanishes as well. Therefore the limit in (21) is negative provided we show

$$\liminf_{n\to\infty} \frac{1}{|\Lambda_n|} \int \ln\left(\frac{Z_{\Lambda_n}(\omega)}{Z_{n+R_0}}\right) dP(\omega) \geq 0. \tag{23}$$

From the definition of $Z_{\Lambda_n}(\omega)$,

$$Z_{\Lambda_n}(\omega) \geq \int \mathbb{1}_{\left\{\omega'_{\Lambda_n\setminus\Lambda_{n-R_0}}=\emptyset\right\}} e^{-H(\omega'_{\Lambda_n})} \pi_{\Lambda_n}(d\omega'_{\Lambda_n}) = e^{-|\Lambda_n\setminus\Lambda_{n-R_0}|} Z_{n-R_0}$$

and therefore

$$\liminf_{n\to\infty} \frac{1}{|\Lambda_n|} \int \ln\left(\frac{Z_{\Lambda_n}(\omega)}{Z_{n+R_0}}\right) dP(\omega) \geq \liminf_{n\to\infty} \frac{1}{|\Lambda_n|} \left(\ln(Z_{n-R_0}) - \ln(Z_{n+R_0}) - |\Lambda_n\setminus\Lambda_{n-R_0}|\right) = 0$$

which proves (23). The proof of Theorem 1 is complete if we show that any stationary probability measure P solving the equality in (9) is a Gibbs measure. We follow essentially the scheme of Preston (1976) (In the variant used in Georgii (1994a), Section 7). So let P be a stationary probability measure such that $\mathcal{I}(P) + H(P) + p_H = 0$. Let us show that for any bounded local function g and any bounded set Λ, $\int g(\omega)P(d\omega) = \int g_\Lambda(\omega)P(\omega)$ where the function g_Λ is defined by

$$g_\Lambda(\omega) = \int g(\omega'_\Lambda \cup \omega_{\Lambda^c}) f_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c}) \pi_\Lambda(\mathrm{d}\omega'_\Lambda).$$

Without loss of generality we assume in the following that $|g|$ is bounded by one. Thanks to the equality (21), for n large enough $\mathcal{I}(P_{\Lambda_n}, Q_n)$ is finite and therefore P_{Λ_n} admits a density with respect to Q_n which we denote by f_n. Let Λ' be a bounded set such that $\Lambda^\oplus \subset \Lambda'$ and such that g is $\mathcal{F}_{\Lambda'}$-measurable. For n large enough such that $\Lambda' \subset \Lambda_n$, the probability measure $P_{\Lambda'}$ admits a density with respect to Q_n restricted to Λ' which we denote by $f_{n,\Lambda'}$. Since $\mathcal{I}(P_{\Lambda_n}, Q_n)/|\Lambda_n| \to 0$, using the standard Lemma 7.5 in Georgii (1994a), for any $\delta > 0$ there exists n large enough and a set Λ' with $\Lambda^\oplus \subset \Lambda' \subset \Lambda_n$ such that

$$\int |f_{n,\Lambda'} - f_{n,\Lambda'\setminus\Lambda}| dQ_n < \delta.$$

We obtain that

$$\int g(\omega) - g_\Lambda(\omega)P(d\omega) = \int f_{n,\Lambda'}(\omega)g(\omega) - f_{n,\Lambda'\setminus\Lambda}(\omega)g_\Lambda(\omega)Q_n(d\omega).$$

From the definition of Q_n and since $\Lambda^\oplus \subset \Lambda_n$ we have

$$\int f_{n,\Lambda'\setminus\Lambda}(\omega)g_\Lambda(\omega)Q_n(d\omega) = \int f_{n,\Lambda'\setminus\Lambda}(\omega)g(\omega)Q_n(d\omega)$$

and we deduce that $|\int g(\omega) - g_\Lambda(\omega)P(d\omega)| \leq \delta$. Letting δ tends to zero we get the DLR equation on Λ. The proof of Theorem 1 is complete.

Acknowledgement: This work was supported in part by the Labex CEMPI (ANR-11-LABX-0007-01)

References

Baddeley, A. J. and M. N. M. V. Lieshout (1995). Area-interaction point processes. *Ann. Inst. Statist. Math. 47*(4), 601–619.

Chiu, S. N., D. Stoyan, W. S. Kendall, and J. Mecke (2013). *Stochastic geometry and its applications* (3 ed.). John Wiley & Sons.

Dereudre, D., R. Drouilhet, and H. Georgii (2012). Existence of Gibbsian point processes with geometry-dependent interactions. *Probability Theory and Related Fields* (Vol. 153, No 3-4), 643–670.

Dereudre, D. and H. Georgii (2009). Variational principle of gibbs measures with delaunay triangle interaction. *Electronic Journal of Probability* (14), 2438–2462.

Dereudre, D. and F. Lavancier (2015). Consistency of likelihood estimation for gibbs point processes. *Preprint, arXiv:1506.02887*.

Georgii, H. (1994a). The equivalence of ensembles for classical systems of particles. *Journal of Statistical Physics 80, Nos 5/6*, 1341–1377.

Georgii, H. (1994b). Large deviations and the equivalence of ensembles for gibbsian particle systems with superstable interaction. *Probability Theory and Related Fields 99*, 171–195.

Georgii, H.-O. (1979). *Canonical Gibbs measures.* Number 760 in Lecture Notes in Mathematics. Springer.

Georgii, H. O. (2011). *Gibbs measure and phase transitions, second edition.* De Gruyter.

Kendall, W. S., M. N. M. V. Lieshout, and A. J. Baddeley (1999). Quermass-interaction processes conditions for stability. *Adv. Appli. Prob. 31*, 315–342.

Landford, O. and D. Ruelle (1969). Observables at infinity and states with short range correlations in statitical mechanics. *Communication in Mathematical Physics 13*, 194–215.

Matthes, K., J. Kerstan, and J. Mecke (1978). *Infinitely Divisible Point Processes* . Chichester: Wiley.

Nguyen, X. and H. Zessin (1979). Integral and differential characterizations Gibbs processes. *Mathematische Nachrichten 88*(1), 105–115.

Preston, C. (1976). *Random fields.* Springer Verlag.

Ruelle, D. (1969). *Statistical Mechanics. Rigorous Results.* (Benjamin, New-York).

Ruelle, D. (1970). Superstable interactions in classical statistical mechanics. *Comm. Math. Phys. 18*, 127–159.

Van Lieshout, M. (2000). *Markov point processes and their applications.* World Scientific.

Widom, B. and J. Rowlinson (1970). New model for the study of liquid-vapor phase transitions. *J. Chem. Phys. 52*, 1670–1684.